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THE 3-MANIFOLD RECOGNITION PROBLEM

ROBERT J. DAVERMAN AND THOMAS L. THICKSTUN

ABSTRACT. We introduce a natural Relative Simplicial Approximation Property for maps from a 2-cell to a generalized 3-manifold and prove that, modulo the Poincaré Conjecture, 3-manifolds are precisely the generalized 3-manifolds satisfying this approximation property. The central technical result establishes that every generalized 3-manifold with this Relative Simplicial Approximation Property is the cell-like image of some generalized 3-manifold having just a 0-dimensional set of nonmanifold singularities.

1. INTRODUCTION

The manifold recognition problem, originally proposed in 1978 by J. W. Cannon [9], asks for a short list of simple topological properties, easy to check, that characterize topological manifolds among topological spaces. Cannon conjectured that n-manifolds might be characterized as those generalized n-manifolds satisfying a minimal amount of general position. To address the latter in dimensions greater than 4 he proposed the following Disjoint Disks Property: any two maps of B^2 into the space can be approximated by maps with disjoint images.

This paper addresses the 3-manifold recognition problem. For that dimension the fundamental difficulty is to identify an appropriate general position property. The Disjoint Disks Property, possessed by no 3-manifold, is impossibly strong, and the related Disjoint Arcs Property, possessed by all generalized 3-manifolds, is impossibly weak.

A *generalized n-manifold* X, abbreviated as n-gm, is a locally compact, locally contractible, finite dimensional metric space with the relative local homology of $\mathbb{R}^n$ (i.e., $H_*(X, X-\{x\}; \mathbb{Z})$ is isomorphic to $H_*(\mathbb{R}^n, \mathbb{R}^n-\{0\}; \mathbb{Z})$ for all $x \in X$). In such a space X the *manifold set*, $M(X)$, consists of all points of X having a neighborhood homeomorphic to $\mathbb{R}^n$, and the *singular set*, or *nonmanifold set*, $S(X)$, is defined as $S(X) = X - M(X)$. As components of locally compact metric spaces are separable, we simply will view all n-gms as separable metric spaces.

Clearly every n-manifold is an n-gm, but the converse fails for $n > 2$. If $f\colon M \to X$ is a proper, cell-like, surjective mapping defined on an n-manifold, where $\dim X < \infty$, then X is an n-gm, and classical examples like the famous dogbone space of R. H. Bing [3] demonstrate that X need not be a genuine manifold. Historically cell-like maps like Bing's have been used to produce a large class of

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examples. To distinguish such images from other possible examples that arise, one calls an n-gm X *resolvable* if there exist a genuine n-manifold M and a proper, cell-like, surjective mapping $f\colon M \to X$. In this case, the pair (M, f) is called a *resolution* of X. Bryant, Ferry, Mio and Weinberger have established the existence of nonresolvable n-gms for $n > 5$ [7].

In dimensions greater than 4 the model theorem is provided by the combination of results by Edwards and Quinn. Given a connected n-gm X, Quinn [19] produced an integer valued obstruction, $i(X)$, which is locally defined, locally constant, and satisfies $i(X \times X') = i(X) \times i(X')$, where $i(X) = 1$ if and only if X is resolvable ($n > 3$). Edwards [13] (see [11] for details) showed that a resolvable n-gm, $n > 4$, is an n-manifold if and only if it satisfies the Disjoint Disks Property. Consequently, for $n > 4$ a connected space X is an n-manifold if and only if X is an n-gm satisfying both the Disjoint Disks Property and $i(X) = 1$.

Daverman and Repovš [12] introduced a kind of general position property—called the spherical simplicial approximation property, abbreviated as SSAP, and defined in Section 4—and showed that every resolvable generalized 3-manifold with the SSAP is a 3-manifold. Here we modify their property, defining a relative simplicial approximation property (RSAP) which is stronger than this SSAP; our main result establishes that, modulo the Poincaré Conjecture, every generalized 3-manifold X satisfying this RSAP is a 3-manifold. Specifically, the fundamental issue is to confirm that X is resolvable, for then [12] applies to give the final 3-manifold recognition step. With no extra hypotheses we produce a cell-like, surjective mapping $\Phi\colon Y \to X$, where Y is a 3-gm such that $S(Y)$ is 0-dimensional. If the Poincaré Conjecture is true, however, then the Corollary to the Resolution Theorem of [22] (see [23] for corrections) assures that Y has a resolution $\Psi\colon M \to Y$, and $\Phi\Psi\colon M \to X$ serves as the desired resolution of X.

2. Preliminaries

A subset C of a space X is *locally k-coconnected*, abbreviated as k-LCC, if each neighborhood U of an arbitrary point $x \in X$ contains another neighborhood V of x such that every map $\partial I^{k+1} \to V - C$ can be extended to a map $I^{k+1} \to U - C$.

We shall distinguish simplicial complexes from their underlying point sets, called *polyhedra*. A *triangulation* of a polyhedron Q is a pair (T, h), where T denotes a simplicial complex and h a homeomorphism of its underlying point set, denoted by $|T|$, onto Q. Frequently the polyhedra encountered here will be subsets of a given 3-gm. One should not presume the existence of any compatibility between the (piecewise) linear structure of the simplicial complex associated to a polyhedron Q in a 3-gm X and the possible linear structures arising within X. Most of our attention will fall on 2-dimensional polyhedra, called 2-polyhedra for short.

A *subpolyhedron* Q' of a polyhedron Q is a closed subset of Q such that there exist a triangulation (T, h) of Q and a subcomplex T' of T with $h(|T'|) = Q'$.

Suppose Q is a polyhedron and $z \in Q$. Impose a triangulation (T, h) on Q. Suppress h, here and throughout the remainder of this paper, and regard T as a simplicial complex whose underlying point set equals Q. Subdivide T, if necessary, so that z corresponds to a vertex of T. For such a Q topologically embedded as a closed subset of a generalized 3-manifold X, $X - Q$ is said to have *free local fundamental group at $z \in Q$*, abbreviated as 1-FLG at z, if for each sufficiently small neighborhood U of z there exists another neighborhood V of z with $z \in V \subset U$

and if W is any connected open set with $z \in W \subset V$, then for each nonempty component W' of $W - Q$ the (inclusion-induced) image $\pi_1(W') \to \pi_1(U')$ is a free group on $m - 1$ generators, where U' denotes the component of $U - Q$ containing W' and m is the number of "components" of $St(z) - z$ whose images meet $Cl(W')$, where $St(z)$ denotes the simplicial star of z in the complex T. As usual, $X - Q$ is simply said to be 1-FLG in X if it is 1-FLG in X at each point of Q.

For simplicity, we will say that a polyhedron Q embedded in a generalized 3-manifold X as a closed subset is *tamely embedded in* X if $X - Q$ is 1-FLG in X. Nicholson [18] has shown that a polyhedron tamely embedded in a genuine 3-manifold M in this 1-FLG sense is tamely embedded in the geometric sense, where there exists a self-homeomorphism (arbitrarily close to *Identity*: $M \to M$) of M that carries Q onto a subspace underlying a subcomplex of some preassigned triangulation of M, after subdivision.

Given maps $\phi\colon Y \to X$ and $f\colon Z \to X$, where X is metrizable, and given $A \subset Y$, we say that f *approximately lifts to* A (occasionally, for emphasis, *under* ϕ) if for each metric on X and each $\epsilon > 0$ there exists a map $\tilde{f}\colon Z \to A$ such that $\phi\tilde{f}$ is within ϵ (pointwise) of f.

Suppose X is a connected 3-gm, D and E are disjoint, closed subspaces of X, and $\mu\colon R \to X - (D \cup E)$ is a map defined on a compact, 2-polyhedron R. We say that μ *homologically separates* D *from* E if there exist $\alpha \in H_2(R;\mathbb{Z}_2)$ and $\xi \in H_3(X - E, X - (D \cup E);\mathbb{Z}_2)$ such that, for each $d \in D$, $\mu_*(\alpha) = i_*\partial(\xi) \not\cong 0$, where i denotes the inclusion $X - (D \cup E) \to X - (\{d\} \cup E)$. We say that μ *strongly separates* D *and* E if no component of $X - \mu(R)$ contains points of both D and E.

A compact subset C of any ANR Y is *cell-like* if, for each open subset U of Y containing C, the inclusion $C \to U$ is homotopic to a constant. A proper map $f\colon Y \to Z$ defined on an ANR Y is a *cell-like map* if each $f^{-1}(z)$, $z \in Z$, is cell-like. We say that a cell-like map $f\colon Y \to Z$ is *conservative over* $B \subset Z$ if $f|f^{-1}(B)$ is 1-1.

Similarly, a compact subset C of an n-manifold M is *cellular* if M contains a sequence $\{D_i\}_{i=1}^{\infty}$ of n-cells such that $Int(D_i) \supset D_{i+1}$ and $\bigcap_{i=1}^{\infty} D_i = C$, and a proper map $F\colon M \to Z$ defined on M is *cellular* if each $F^{-1}(z)$ is.

As in [22] a 3-*near* manifold M^* is a 3-gm obtained from a 3-manifold M by identifying a null sequence of pairwise disjoint 3-cells in M and replacing the interior of each with the interior of a compact, contractible 3-manifold in such a way that $S(M^*)$ is 0-dimensional and 1-LCC embedded in M^*. A *near resolution* of a 3-gm X is a pair (M^*, ψ), where M^* is a 3-near manifold and $\psi\colon M^* \to X$ is a proper, cell-like surjection. Should the Poincaré Conjecture be false, one could easily produce a 3-near manifold M^* which is nonresolvable, homotopy equivalent to S^3, has $S(M^*) = point$, and satisfies $M^* \times \mathbb{R} \cong S^3 \times \mathbb{R}$.

3. Elementary properties of 3-gms and 3-near manifolds

A *generalized* 3-*manifold with boundary* Z is a locally compact, locally contractible, finite dimensional metric space such that, for each $z \in Z$, either $H_*(Z, Z - \{z\};\mathbb{Z}) \cong 0$ or $H_*(Z, Z - \{z\};\mathbb{Z}) \cong H_*(\mathbb{R}^3, \mathbb{R}^3 - \{0\};\mathbb{Z})$; the subset consisting of all $z \in Z$ for which $H_*(Z, Z - \{z\};\mathbb{Z}) \cong 0$ is called the *boundary of* Z, denoted ∂Z.

Lemma 3.1. *If the space* Z *is expressed as a union of closed subsets* Z_1 *and* Z_2 *of* Z *which are generalized* 3*-manifolds with boundary, where* $Z_1 \cap Z_2 = \partial Z_1 = \partial Z_2$*,* *then Z is a generalized 3-manifold. Conversely, if Z is a 3-gm and $Z = Z_1 \cup Z_2$, where Z_1, Z_2 are closed subsets of Z and $Z_0 = Z_1 \cap Z_2$ is a 2-gm, then Z_1 and Z_2 are generalized 3-manifolds with boundary.*

Proof. For the most part—except for ANR properties—this is treated in [20]. When Z_1 and Z_2 are 3-gms with boundary, work of Mitchell [16] combines with classical results of Wilder [24] to establish that ∂Z_i $(i = 0, 1)$ is a 2-manifold, hence an ANR, and standard results from ANR theory then yield that $Z = Z_1 \cup Z_2$ is an ANR. Similarly, in the converse, Z_0 is an ANR, so Z_1 and Z_2 must be ANRs as well. □

Lemma 3.2. *Let $\{X_i, p_{i+1,i}\}$ denote a sequence of 3-gms and cell-like maps, with inverse limit Z. Then Z is a 3-gm, and the associated projections $q_i\colon Z \to X_i$ are cell-like maps.*

Proof. We provide an argument only for the case in which each X_i is a manifold factor, i.e., $X_i \times \mathbb{R}^k$ is a manifold (one can take k to be any fixed integer greater than 1). It parallels the proof of [17, 3.9(iii)] about the inverse limit of a sequence of ANRs and cell-like maps yielding an ANR. Only this special case matters for our purposes here, because the RSAP implies X contains 2-cells, so $X \times \mathbb{R}^k$ contains codimension one cells, and thus the Quinn obstruction [19] to the existence of a resolution vanishes.

Examine the related sequence $\{X_i \times \mathbb{R}^k, p_{i+1,i} \times Id\}$ of cell-like maps between manifolds. By [13] or [21] each map $p_{i+1,i} \times Id$ is a near-homeomorphism, so a result of M. Brown [6] (or see [1]) assures that the induced limiting map $q_1 \times Id\colon Z \times \mathbb{R}^k \to X_1 \times \mathbb{R}^k$ is a near-homeomorphism. Hence, $Z \times \mathbb{R}^k$ is a $(3 + k)$-manifold, and Z, being one of its codimension k factors, must be a 3-gm. Furthermore, $q_1 \times Id$, being a near-homeomorphism, is a cell-like mapping [11, Theorem 17.4]; obviously this means q_1 itself is cell-like. □

The next lemma uses the notation of Lemma 3.2, as well as the standard notation for the composite, $p_{2,1} \cdots p_{k,k-1} p_{k+1,k} = p_{k+1,1}$. The map q_1 is the inverse limit projection described in Lemma 3.2.

Lemma 3.3. *Let $\{X_i, p_{i+1,i}\}$ denote a sequence of 3-gms and cell-like maps such that $p_{k+1,k}$ restricts to a cellular map $p^{-1}_{k+1,1}(M(X_1)) \to p^{-1}_{k,1}(M(X_1))$ for each $k > 0$. Then $q_1^{-1}(M(X_1))$ is a 3-manifold.*

Proof. Each of the restricted $p_{k+1,k}$ is a near-homeomorphism by Armentrout's Cellular Approximation Theorem [2], so Brown's argument [6] applies, just as in 3.2. □

Throughout the remainder of this section $\mathbb{Z}_2$ coefficients will be used for all homology and cohomology computations.

Lemma 3.4. *Suppose E is a nonempty, closed subset of the 3-gm X and $d \in X - E$. Then there exist a compact, connected neighborhood D of d, a compact 2-polyhedron R, and a map $\nu\colon R \to X$ such that ν homologically separates D from E.*

Proof. Note that whenever X has no compact component,

$$\partial\colon H_3(X, X - \{x\}) \to H_2(X - \{x\})$$

is 1-1. This follows immediately, because, by duality [5], $H_3(X) \cong H^0_c(X) \cong 0$. Fix $0 \neq \xi \in H_3(X - E, X - (E \cup \{d\}))$, and assume X is connected (so $X - E$

has no compact component). Simply choose R and $\nu\colon R \to X$ to be a carrier of $\partial\xi(\ncong 0) \in H_2(X-(E\cup\{d\}))$; this choice assures that ν homologically separates $\{d\}$ from E.

Fix a compact connected neighborhood $D \subset X-(E\cup\mu(R))$ of d. Given any $d' \in D$, find an arc $\gamma \subset X-(E\cup\mu(R))$ joining d to d'. The bottom level map in the diagram below is an isomorphism, since all the others are (the vertical ones, by duality in $X-E$):

$$\begin{array}{ccc} H^0(\gamma) & \longrightarrow & H^0(\{d\}) \\ \downarrow & & \downarrow \\ H_3(X-E, X-(E\cup\gamma)) & \longrightarrow & H_3(X-E, X-(E\cup\{d\})) \end{array}$$

Similarly,

$$H_3(X-E, X-(E\cup\gamma)) \to H_3(X-E, X-(E\cup\{d'\}))$$

is an isomorphism. It follows that ν homologically separates D from E. □

Lemma 3.5. *Suppose D, E are disjoint, closed subsets of the 3-gm X and $\nu\colon R \to X$ is a map of a compact 2-polyhedron R which homologically separates D from E. Then ν strongly separates D and E.*

Proof. If $\nu(R)$ failed to separate $d_0 \in D$ and $e_0 \in E$, then there would be an arc $\gamma \subset X-\nu(R)$ connecting d_0 and e_0. By hypothesis there exist $\alpha \in H_2(R)$ and $\xi(\neq 0) \in H_3(X-E, X-(E\cup D))$ such that $\nu_*(\alpha) = i_*\partial(\xi)(\neq 0) \in H_2(X-(E\cup\{d_0\}))$. Name a compact carrier $C \subset X-E$ for ξ. Then the image of ξ in $H_3(X-E, X-(E\cup\{d_0\}))$ is nonzero and belongs to the inclusion-induced image

$$\eta_*\colon H_3(C, C\cap(X-(E\cup D))) \to H_3(X-E, X-(E\cup\{d_0\})).$$

Let $\tilde\gamma$ denote the component of $\gamma - E$ containing d_0. Certainly here η_* would factor through

$$H_3(X-E, X-(E\cup\tilde\gamma)) \cong H^0_c(\tilde\gamma) \cong H^0_c([0,1/2)) \cong 0,$$

a contradiction. □

Lemma 3.6. *Let C be a closed subset of a 3-manifold M, the frontier of which is a surface S. Then attachment of an open collar $S\times[0,1)$ to C along $S = S\times 0$ yields a 3-manifold.*

Proof. When M is a 3-sphere and S is a 2-sphere this was proved by Hosay and Lininger [14] (or see [10], [8]). The general case, which localizes to that of a 2-sphere in S^3 [4, Theorem 5], follows. □

4. A relative simplicial approximation property

According to [12], a generalized 3-manifold X has the *Simplicial Approximation Property* (SAP) if for each map $f\colon I^2 \to X$ and each $\epsilon > 0$, there exist a map $F\colon I^2 \to X$ and a compact 2-polyhedron $K_F \subset X$ such that (1) $\mathrm{dist}(F,f) < \epsilon$, (2) $F(I^2) \subset K_F$, and (3) $X - F(I^2)$ is 1-FLG in X. Similarly, X has the *Spherical Simplicial Approximation Property* (SSAP) if the analogous conditions hold for maps $S^2 \to X$ in place of maps $I^2 \to X$.

We will say that a map $f\colon K \to X$ of a compact 2-dimensional polyhedron K to a generalized 3-manifold X is *simplicial* if $f(K)$ is a polyhedron whose complement is 1-FLG in X and $f\colon K \to f(K)$ is simplicial with respect to some triangulations of K and $f(K)$. Of course, given any map between polyhedra, we can impose triangulations, take fine mesh subdivisions, and then approximate by a simplicial map. In short, the map F in the SAP (similarly, in the SSAP) can be assumed to be simplicial and onto K_F.

A generalized 3-manifold X has the *Relative Simplicial Approximation Property* (RSAP) if for each map $f\colon I^2 \to X$, each compact subpolyhedron Q of I^2 for which $f|Q$ is simplicial as above, and each $\epsilon > 0$, there exists a simplicial map $F\colon I^2 \to X$ such that $\operatorname{dist}(F, f) < \epsilon$ and $F|Q = f|Q$.

Lemma 4.1. *Every* 3*-gm X that satisfies the RSAP also satisfies the following stronger property*: *for each compact* 2*-polyhedron K, compact subpolyhedron L, map $g\colon K \to X$ such that $g|L$ is simplicial, and $\epsilon > 0$, there exists a simplicial map $G\colon K \to X$ with* $\operatorname{dist}(G, g) < \epsilon$ *and $G|L = g|L$.*

Proof. Assume for simplicity that X is path connected. List the large simplexes $\Delta_1, \dots, \Delta_r$ of L—large in the sense of being proper faces of no other simplexes of L—and choose any simplex Δ_{r+1} of $K - L$. We show how to approximate g by a new map $g_{r+1}\colon K \to X$ which is simplicial on a complex underlying $L \cup \Delta_{r+1}$.

Specify a finite collection $\sigma_1, \dots, \sigma_r, \sigma_{r+1}$ of pairwise disjoint simplexes in $Int(I^2)$ and equip them with simplicial isomorphisms $e_j\colon \sigma_j \to \Delta_j$ $(j = 1, \dots, r+1)$. Define $\eta = \bigcup e_j\colon \bigcup \Delta_j \to K$. Think of $e_{r+1}^{-1}(\Delta_{r+1} \cap L)$ together with all the other σ_j $(j = 1, \dots, r)$ as $Q \subset I^2$. Use the hypothesized path connectedness of X to extend $g\eta|Q$ to a map $f\colon I^2 \to X$. Apply the RSAP to approximate $f\colon I^2 \to X$ by a simplicial map $F\colon I^2 \to X$ that agrees with $g\eta$ on Q, and define $G_{r+1}\colon Q \cup \Delta_{r+1} \to X$ as $G_{r+1} = F\eta^{-1}$. Note that G_{r+1} is a well-defined simplicial map approximating $g|L \cup \Delta_{r+1}$ and coinciding with g on L. By a controlled homotopy extension lemma, G_{r+1} extends to a map $g_{r+1}\colon K \to X$ approximating g and coinciding with g on L.

A finite number of repetitions of this procedure yields the desired simplicial map $G\colon K \to X$. □

Corollary 4.2. *Every generalized* 3*-manifold X satisfying the RSAP also satisfies the SSAP.*

Corollary 4.3. *All resolvable generalized* 3*-manifolds satisfying the RSAP are genuine* 3*-manifolds.*

See Recognition Theorem 3.1 of [12].

Corollary 4.4. *Suppose X is a* 3*-gm satisfying the RSAP, $L \subset X$ is a tamely embedded* 2*-polyhedron, and $\nu\colon R \to X$ is a map defined on a compact* 2*-polyhedron. Then for each $\epsilon > 0$ there exists a simplicial map $\mu\colon R \to X$ with* $\operatorname{dist}(\mu, \nu) < \epsilon$ *and $L \cup \mu(R)$ is a polyhedron tamely embedded in X.*

We say that a 2-polyhedron P is *preferred* if it contains neither isolated points nor local cut points—equivalently, if in some (hence, each) triangulation of P the link of every vertex is nonempty and connected. More is said about the role of preferred 2-polyhedra in Section 5. For brevity we call a pair (K, P) of compact, 2-polyhedra in a 3-gm X a *tame-preferred polyhedral pair* if K is tame, P is preferred and P is a subpolyhedron of K. Note that if (K, P) is tame-preferred in X, P is not tame—at least, not *a priori* tame—in X.

Lemma 4.5. *Suppose X is a 3-gm satisfying the RSAP and $f\colon I^2 \to X$ is a map such that f restricts to a simplicial map on ∂I^2 with $f|\partial I^2 - I \times 1$ 1-1 and $f(I\times 0) \cap f(I\times 1) = \varnothing$. Then there exists a tame-preferred polyhedral pair (K, P) such that $K \supset P \supset f(I \times 0)$. Furthermore, if $f(\partial I^2 - I \times 1)$ is a subpolyhedron of a compact, tame polyhedron Q, then (K, P) can be obtained so $P \cup Q$ is a subpolyhedron of K.*

Proof. Apply RSAP to obtain an approximation $F\colon I^2 \to X$ to f, with $F|\partial I^2 - I \times 1 = f|\partial I^2 - I \times 1$, and where $F\colon I^2 \to F(I^2)$ can be regarded as simplicial (also, if need be, where $F(I^2) \cup Q$ is a tame 2-polyhedron). Choose triangulations T of I^2 and T' of $F(I^2)$ for which F is simplicial.

Fix a 1-simplex τ of T', $\tau \subset f(I \times 0)$. We show that some 2-simplex $\sigma \in T'$ contains τ. To see why, consider the unique 2-simplex $\gamma \in T$ containing $f^{-1}(\tau)$. Set $\sigma = F(\gamma)$ if $F(\gamma) \neq \tau$. Otherwise, produce a maximal chain $\gamma = \gamma_0, \gamma_1, \ldots, \gamma_s$ of 2-simplexes in T such that $\gamma_{j-1} \cap \gamma_j =$ *edge* and $\gamma_j \subset F^{-1}(\tau)$. Since $\partial\gamma_s - \gamma_{s-1} \not\supset \partial I^2$, some other 2-simplex ξ must meet γ_s in an edge $e = f^{-1}(\tau)$, and $F(\xi) \in T'$ will be a 2-simplex containing τ.

Let v be a vertex of $f(I \times 0)$ and w, w' the two possible points in the link of v there. Essentially the same argument shows that w, w' belong to a single component of the link of v in $F(I^2)$.

Although $F(I^2)$ itself might not be preferred, we claim that it contains a preferred polyhedron $P \supset f(I \times 0)$. Let P' be $F(I^2)$ after deletion of (the interiors of) all those 1-simplexes e of T' which are edges of no 2-simplex from T'. Clearly then $F(I^2) \supset P' \supset f(I \times 0)$. If the vertex $w \in T'$ has disconnected link in P' and $w \notin f(I \times 0)$, delete a small regular neighborhood of w from P'; if, however, $w \in f(I \times 0)$, then delete that small neighborhood $N(w)$ but reinsert the closure of the unique component of $N(w) - \{w\}$ containing the intersection of $N(w)$ with $\mathit{Link}(w, f(I \times 0))$. Repetition of these two operations eliminates or repairs all disconnected links and yields a preferred polyhedron $P \subset P'$ such that P and $P \cup Q$ are subpolyhedra of $K = F(I^2) \cup Q$. □

Lemma 4.6. *Suppose X is a 3-gm satisfying the RSAP and $L \subset X$ is a compact 2-polyhedron tamely embedded in X such that each vertex of L belongs to at least two edges. Then there exists a tame-preferred polyhedral pair (K, P) in X such that L is a subpolyhedron of P.*

Proof. Since components of L can be treated one after another, we will simply assume L is connected.

Assume τ is a 1-simplex of L which belongs to no 2-simplex. In view of the hypothesis here, there is an embedding $f\colon \partial I^2 - I \times 1 \to L$ with $f(I \times 0) = \tau$. Since no arc locally separates a 3-gm, obviously f can be extended to a map $f\colon I^2 \to X$ with $f(I\times 0) \cap f(I \times 1) = \varnothing$, and then Lemma 4.5 assures that L can be expanded by attaching a preferred polyhedron that contains τ. Repeating as often as necessary, we can simply assume each 1-simplex of L is a face of some 2-simplex.

Now assume $v \in L$ is a vertex that has disconnected link in the expanded L'. One can build an embedding $f\colon \partial I^2 - I \times 1 \to L'$ with $v \in f(I \times 0)$, extend f to all of ∂I^2, as before, and apply Lemma 4.5 to reduce the number of components of $\mathit{Link}(v, L')$ in the expanded L. This expansion can be localized to affect none of the other vertices of L. One can eliminate any 1-simplex contained in no 2-simplex from the expansion and snip at new vertices to prevent disconnected links, just as in the proof of 4.5. Finitely many repetitions yields a preferred polyhedron containing all of L. □

Let L denote a 2-polyhedron. Call $v \in L$ a *negligible vertex* if there exists a homeomorphism θ from $[0,1)$ onto a neighborhood of v such that $\theta(0) = v$. Note that no point of a preferred 2-polyhedron is a negligible vertex.

Essentially the same argument as in 4.6 proves the following.

Lemma 4.7. *Suppose X is a 3-gm satisfying the RSAP and $L \subset X$ is a compact 2-polyhedron tamely embedded in X. Let L^* be a compact, polyhedral subset of L obtained by deleting a small connected neighborhood about each negligible vertex of L. Then there exists a tame-preferred polyhedral pair (K, P) in X with L^* a subpolyhedron of P.*

Theorem 4.8. *Suppose X is a 3-gm satisfying the RSAP, D and E are disjoint closed subsets of X, $\nu\colon R \to X$ is a map defined on a compact 2-polyhedron R such that ν homologically separates D and E, and P is a preferred 2-polyhedron tamely embedded in X. Then there exists a map $\mu^*\colon R \to X$ such that μ^* homologically separates D and E and there exists a tame-preferred polyhedral pair (K^*, P^*) in X such that $P^* \supset P \cup \mu^*(R)$.*

Proof. First apply Corollary 4.4 to approximate ν by a simplicial map $\mu\colon R \to X$ so close to ν that μ homologically separates D and E and, in addition, $P \cup \mu(R)$ is a 2-polyhedron. Then use Lemma 4.7 with $L = P \cup \mu(R)$ to obtain a tame-preferred polyhedral pair (K^*, P^*) in X, with $P^* \supset L^*$. Note that any negligible vertex of $P \cup \mu(R)$ must lie in $\mu(R) - P$, so $P^* \supset P$. By construction the map μ, considered as a map to $\mu(R) \subset L$, is homotopic in $\mu(R)$ to a map μ^* into L^*. Hence, D and E are homologically separated by μ^*, and $P \cup \mu^*(R) \subset L^* \subset P^*$. □

5. The main result

The aim of this section is to establish the following Near-Resolution Theorem. It immediately yields the promised characterization of 3-manifolds as the generalized 3-manifolds satisfying the RSAP, provided the Poincaré Conjecture holds.

Theorem 5.1 (Near-Resolution)**.** *Every generalized 3-manifold X satisfying the RSAP has a 3-near resolution (M, ψ).*

Corollary 5.2. *Suppose the Poincaré Conjecture is true. Then a generalized 3-manifold X is a genuine 3-manifold if and only if it satisfies the RSAP.*

Proof. When X satisfies the RSAP, Theorem 5.1 certifies the existence of a cell-like, surjective map $\psi\colon M \to X$ defined on a 3-near manifold M. Under the assumption that the Poincaré Conjecture is true, M actually is a 3-manifold; in other words, the promised cell-like mapping ψ itself provides a resolution of X. Corollary 4.3 confirms that X is a 3-manifold.

The forward implication is trivial. □

Lemma 5.3 (Inflation)**.** *Suppose X is a 3-gm and $P \subset X$ is a preferred 2-polyhedron. Then there exist a 3-gm Y and a proper, surjective, cell-like map $\phi\colon Y \to X$ satisfying the following conditions*:

(1) *ϕ is conservative over $X - P$*;

(2) *there is a preferred 2-polyhedron $\widetilde{P} \subset M(\phi^{-1}(P))$ for which $\phi|\colon \widetilde{P} \to P$ is cell-like*;

(3) *for each* (*respectively, preferred*) 2*-polyhedron* $J \supset P$*, there is a* (*respectively, preferred*) 2*-polyhedron* J^**,* $J^* \subset \phi^{-1}(J)$*, for which* $\phi|\colon J^* \to J$ *is cell-like*;

(4) *for each* $x \in X$*,* $\phi^{-1}(x) \cap S(Y)$ *is finite*; *and*

(5) $\phi^{-1}(M(X)) \subset M(Y)$ *and* $\phi|\colon \phi^{-1}(M(X)) \to M(X)$ *is cellular.*

Proof. We start by describing a model situation in which P is a compact, connected 2-manifold separating X into two components, X_+ and X_-. Here $FrX_+ = P = FrX_-$. Let Y be the space resulting from the disjoint union of ClX_-, $P \times [-1,1]$ and ClX_+ after identifying each $x \in FrX_-$ with $x \times -1 \in P \times [-1,1]$ and each $x \in FrX_+$ with $x \times 1 \in P \times [-1,1]$. Define $\phi\colon Y \to X$ as the obvious map induced by inclusions on the images of ClX_-, ClX_+, extended to send all of $z \times [-1,1]$, $z \in P$, to $z \in P \subset X$. Lemma 3.1 assures that Y is a generalized 3-manifold. One can check quite easily that $\phi\colon Y \to X$ has all the right features. In particular, the (preferred) 2-polyhedron $\widetilde{P}$ called for in (2) can be spelled out as $\widetilde{P} = P \times \{0\} \subset P \times [-1,1]$, and the polyhedron J^* called for in (3) can be defined as

$$J^* = \widetilde{P} \cup \phi^{-1}(J - P) \cup [P \cap Cl(J - P)] \times [-1,1].$$

Conclusion (4) is obvious. Conclusion (5) is assured by Lemma 3.6. Finally, since each point preimage is a cell, cellularity of ϕ over $M(X)$ is guaranteed here, as well as in subsequent steps, by [15, Cor. 1.4] and [11, Prop. 18.4].

Impose a triangulation T on P. Locally the same procedure as in the model case works at interiors of all 2-simplexes $\sigma \in T$ and leads to a cell-like map $\phi_2\colon Y_2 \to X$ defined on a 3-gm Y_2. When replacing $Int(\sigma)$ by $Int(\sigma) \times [-1,1]$, σ a 2-simplex of T, the topology of Y_2 must be regulated so that given any sequence $\{p_n\}$ in $Int(\sigma)$ converging to $p_0 \in \partial\sigma$, then $p_n \times [-1,1] \to p_0$.

The next step is to inflate the 1-skeleton $T^{(1)}$ of T, treated as a subset of Y_2, to put it in the manifold set of another 3-gm Y_1. At each 1-simplex $\tau \in T$, whereas $Int(\tau)$ has a neighborhood V_τ in X whose structure is represented schematically in Figure 1(a), the neighborhood $\phi_2^{-1}(V_\tau)$ in Y_2 has structure represented in Figure 1(b). This is the spot where the value of preferred 2-polyhedron is exposed. Each $\tau \in T^{(1)}$ is a face of 2-simplexes $\sigma_1, \sigma_2, \dots, \sigma_m$, $m \geq 1$, in T; we presume these are arranged in a circular order, in the sense that both σ_j and σ_{j+1} ($j = 1, 2, \dots, m$; $j+1$ understood to be 1 when $j = m$) meet the frontier of some component W_j of $V_\tau - P$. With care in the construction of V_τ, we can assure that $W_1, W_2, \dots, W_m$ constitute all the components of $V_\tau - P$.

The only significant difference between the structures in X or Y_2 and the schematics is that $Int(\tau)$ is an open interval, not just the special point in schematics. The segments emanating from that point in Figure 1 also must be enlarged by taking Cartesian products with that open interval, regarded as $Int(\tau)$.

In place of each $Int(\tau)$ we will insert $Int(\tau) \times B^2$ into Y_2 to form a new 3-gm Y_1 (topologized like Y_2) and cell-like map $\phi_1\colon Y_1 \to Y_2$, one which is conservative over $(Y_2 - |T^{(1)}|) \cup |T^{(0)}|$. Specifically, replace $Y_2 - |T^{(0)}|$ with the space obtained from the disjoint union of $Int(\tau) \times B^2$, thickened 2-simplexes $\sigma_i \times [-1,1]$ and closures ClW_j of components of the various $\phi_2^{-1}(V_\tau - P)$ by attaching $Int(\tau) \times [-1,1] \subset \sigma_i \times [-1,1]$ to an arc of $Int(\tau) \times \partial B^2$, as shown in Figure 2, and (localized) by attaching ClW_j to

$$Int(\sigma_j) \times [-1,1] \cup Int(\sigma_{j+1}) \times [-1,1] \cup Int(\tau) \times \partial B^2$$

via the map sending $z \in ClW_j \cap (\sigma_j - \tau)$ to $z \times 1 \subset Int(\sigma_j) \times [-1,1]$ and sending $z \in ClW_j \cap (\sigma_{j+1} - \tau)$ to $z \times -1 \subset Int(\sigma_{j+1}) \times [-1,1]$. The cell-like map ϕ_1

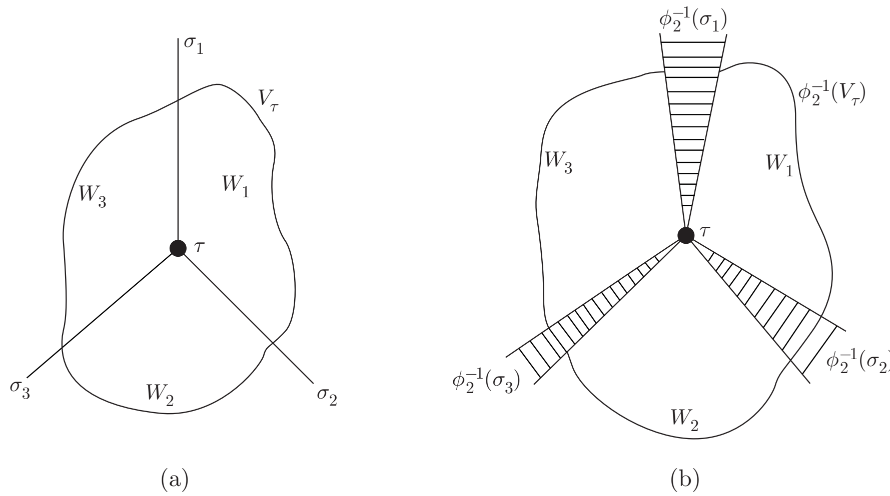

FIGURE 1.

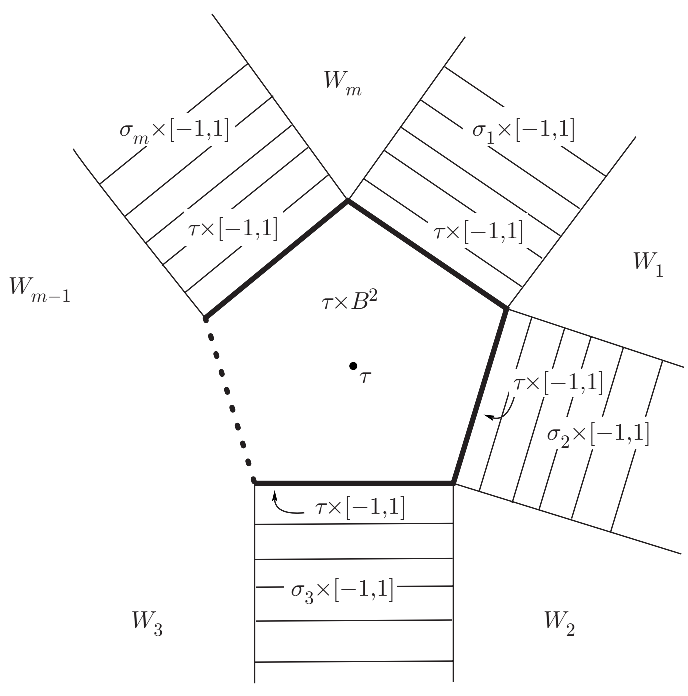

FIGURE 2.

amounts to first coordinate projection $Int(\tau) \times B^2 \to Int(\tau)$ wherever that makes sense; elsewhere it is conservative. Let $\widetilde{P_2}$ denote the preferred 2-polyhedron of (2) obtained in Y_2, and let P_τ denote the product of $Int(\tau)$ with the segments in B^2

associated with τ; then the preferred 2-polyhedron $\widetilde{P_1}$ of (2), contained, except for its 0-skeleton, in $M(Y_1)$, is the closure of $\phi_1^{-1}(\widetilde{P_2} - |T^{(1)}|) \cup (\bigcup_\tau P_\tau)$. The J^* at this stage are defined similarly. Note that each nontrivial point preimage under ϕ_1 meets $S(Y_1)$ in a finite set.

When $y_1 \in Int(\tau) \cap M(Y_2)$, $\tau \in T^{(1)}$, one can find a small neighborhood N of y_1 such that $\phi_1^{-1}(N)$ can be expressed as $(B^+ \times \mathbb{R}) \cup (\bigcup_i Cl(V_i))_{i=1}^s$, where $\{V_i\}$ is the collection of components of $N - \widetilde{P_2}$ and B^+ is a disk B to which s open collars on arcs $\alpha_1, \ldots, \alpha_s$ are attached, where $\alpha_i \cap \alpha_{i+1} = a_i$ and $Cl(V_i) \cap (B \times \mathbb{R}) = a_i \times \mathbb{R}$. Repeated applications of Lemma 3.6, one for each component V_i, yields that $\phi_1^{-1}(M(Y_2))$ is a 3-manifold.

Finally we must blow up vertices $v \in T^{(0)}$ to 3-cells B_v. Each v has a well-defined link L_v in $P \subset X$ and a thickening $T_v = (\phi_2\phi_1)^{-1}(L_v)$ of a copy of L_v, namely, $T_v \cap \widetilde{P_1}$, to a compact 2-manifold with boundary. We argue that T_v embeds in a 2-sphere S_v.

Claim. The space S_v obtained by attaching a disk to each component of ∂T_v is a 2-sphere.

Proof of the Claim. There is a closed neighborhood C_v of v in Y_1 that meets $(\phi_2\phi_1)^{-1}(P)$ in a subset homeomorphic to the cone (from v) over T_v. Set $C_v^- = C_v \cap (\phi_2\phi_1)^{-1}(P)$. Replace the various component closures $Z_1, \ldots, Z_t$ of $C_v - C_v^-$ by 3-cells $B_1, \ldots, B_t$ with

$$B_j \cap C_v^- = Z_j \cap C_v^- = coneJ \ (= \text{2-cell}),$$

J representing a component of ∂T_v, to form $Q_v = C_v^- \cup \bigcup_j B_j$. Note that Q_v can be regarded as the cone from v over S_v, where S_v denotes T_v capped off with 2-cells, one in each $\partial B_j - \{v\}$. Clearly S_v is a 2-manifold. Moreover, each 1-cycle $[z]$ in $S_v - \{v\}$ is homologous to one in $T_v \subset S_v$. Since loops in T_v can be deformed in $X_v - \{v\}$ arbitrarily close to v and since $Int(C_v)$ is a 3-gm, each such loop λ is null homologous in $C_v - \{v\}$. In view of the fact that the various $B_j - \{v\}$ are absolute extensors, the inclusion $C_v^- - \{v\} \to Q_v - \{v\}$ factors through $C_v - \{v\}$. It follows that each λ is null-homologous in $Q_v - \{v\} \approx S_v \times [0, 1)$. Hence, $H_1(S_v) \cong 0$, and S_v must be a 2-sphere, which completes the proof of the Claim.

Continuing with the proof of 5.3, we regard S_v as the boundary of a 3-cell B_v, replace each $v \in T^{(0)}$ with B_v in a new 3-gm Y_0, and define $\phi_0\colon Y_0 \to Y_1$ as the map sending each B_v to the associated vertex v and being conservative over the complement of the 0-skeleton $T^{(0)}$. Here the S_v of the Claim is modified by identifying each of the (abstractly) attached disks to points, which does not change S_v topologically. It has the benefit of providing a finite set $F_v \subset S_v$ such that $S_v - F_v$ has a neighborhood which meets $\phi_0^{-1}(Y_1 - |T^{(0)}|)$ in a 3-manifold thickening of $\phi_0^{-1}(\widetilde{P_1} - |T^{(0)}|)$. The topology near B_v can be arranged so that the closure of $\phi_0^{-1}(\widetilde{P_1} - |T^{(0)}|)$ meets S_v in a 1-dimensional polyhedron K_v. Again each $\phi_0^{-1}(z) \cap S(Y_0)$ is finite, and K_v is a strong deformation retract of $S_v - F_v$. The final $\widetilde{P} = \widetilde{P_0}$ is $\phi_0^{-1}(\widetilde{P_1} - |T^{(0)}|) \cup (\text{cone over } K_v)$; the final J^* is obtained similarly.

The desired map will be $\phi = \phi_2\phi_1\phi_0$. As in Lemma 3.3, it is a near-homeomorphism over $M(X)$, so its retraction to $\phi^{-1}(M(X)) \subset M(Y)$ is cellular [11, Prop. 5.1]. □

The map $\phi\colon Y \to X$ in the conclusion of the preceding lemma will be called an *inflation of X at K*.

Lemma 5.4. *Suppose X is a 3-gm satisfying the RSAP. Then there exists a sequence $\{K_i, P_i\}_{i\geq 1}$ of tame-preferred polyhedral pairs in X, with $P_i \subset P_{i+1}$ for all $i \geq 1$, and there exists a sequence of maps $\mu_i\colon R_i \to X$ defined on compact 2-polyhedra R_i, with $\mu_i(R_i) \subset P_i$ for all $i \geq 1$, such that corresponding to any two points $x, x' \in X$ is an index $k \in \mathbb{N}$ for which μ_k homologically separates x from x' in X.*

Proof. Being treatable componentwise as a separable metric space, by an initial assumption, X has a countable basis Ω. Enumerate the countable collection of pairs $\Lambda = (W_j, W_j')_{j=1}^{\infty} \in \Omega \times \Omega$ for which $Cl(W_j) \subset W_j'$ and some map $\nu_j\colon R_j \to X$, defined on a compact 2-polyhedron R_j, homologically separates $Cl(W_j)$ from $X - W_j'$. Lemma 3.4 assures that for any two points $x, x' \in X$ there is a pair $(W_j, W_j') \in \Lambda$ with $x \in W_j$, $x' \in X - W_j'$.

Since X satisfies RSAP, Theorem 4.8 provides a tame-preferred polyhedron pair (K_1, P_1) in X and a map $\mu_1\colon R_1 \to X$ with $\mu_1(R_1) \subset P_1$, such that μ_1 homologically separates $Cl(W_1)$ and $X - W_1'$.

Assume that we have already produced a finite collection of tame-preferred polyhedral pairs $(K_1, P_1), (K_2, P_2), \ldots, (K_t, P_t)$ in X with

$$P_1 \subset P_2 \subset \cdots \subset P_t$$

and maps $\mu_j\colon R_j \to X$ with $\mu_j(R_j) \subset P_j$ and with μ_j strongly separating $Cl(W_j)$ and $X - W_j'$ $(j = 1, 2, \ldots, t)$. Again Theorem 4.8 provides a tame-preferred polyhedral pair (K_{t+1}, P_{t+1}) in X with $P_{t+1} \supset P_t$ and a map $\mu_{t+1}\colon R_{t+1} \to X$ with $\mu_{t+1}(R_{t+1}) \subset P_{t+1}$ such that μ_{t+1} homologically separates $Cl(W_{t+1})$ and $X - W_{t+1}'$. □

Lemma 5.5 (Resolution). *Suppose the 3-gm X contains a sequence $\{P_i\}_{i=1}^{\infty}$ of compact, preferred 2-polyhedra such that $P_i \subset P_{i+1}$ for all $i \geq 1$. Then there exist a 3-gm Y and a proper, cell-like, surjective map $\Phi\colon Y \to X$ satisfying the following conditions*:

(i) *every map $\mu\colon R \to P_k$, $k \in \mathbb{N}$, defined on a compact 2-polyhedra R has approximate lifts into $M(Y)$, and*

(ii) *for each $p \in X$, $\dim[\Phi^{-1}(p) \cap S(Y)] \leq 0$.*

Proof. Set $X_1 = X$ and $\{P_i^{(1)} = P_i\}_{i=1}^{\infty}$. By induction we will construct, for each $n \in \mathbb{N}$, a proper, cell-like map $\phi_{n+1,n}\colon X_{n+1} \to X_n$ together with a certain sequence, $\{P_i^{n+1}\}_{i=n+1}^{\infty}$, of compact, preferred 2-polyhedra in X_{n+1}. The desired map $\Phi\colon Y \to X$ will be the inverse limit of the inverse sequence of maps $\{X_n, \phi_{n+1,n}\}$.

Apply Inflation Lemma 5.3 to obtain an inflation $\phi_{2,1}\colon X_2 \to X_1$ at $P_1^{(1)} = P_1$. Among other features, this provides a 2-polyhedron $\widetilde{P_1} \subset M(\phi_{2,1}^{-1}(P_1)) \subset X_2$ where $\phi_{2,1}|\colon \widetilde{P_1} \to P_1$ is cell-like. Let $\{P_i^{(2)}\}_{i=2}^{\infty}$ be approximate lifts of P_i described in conclusion (3) there. Assuming cell-like maps $\phi_{n+1,n}\colon X_{n+1} \to X_n$ defined on 3-gms X_{n+1} have been obtained for $n = 1, 2, \ldots, t$, along with approximate lifts $\{P_i^{(n+1)}\}_{i=n+1}^{\infty}$ of $\{P_i^{(n)}\}_{i=n+1}^{\infty}$, and 2-polyhedra $\widetilde{P_n} \subset M(\phi_{n+1,n}^{-1}(P_n^{(n)})) \subset X_{n+1}$ for which $\phi_{n+1,n}|\colon \widetilde{P_n} \to P_n^{(n)}$ is cell-like, apply Inflation Lemma 5.3 again to obtain an inflation of X_{n+1} at $P_{n+1}^{(n+1)}$, thereby producing the next level of objects for $n = t + 1$.

We conclude immediately from Lemma 3.2 that the inverse sequence $\{X_n, \phi_{n+1,n}\}$ has inverse limit $\Phi\colon Y \to X_1 = X$, with Y a 3-gm and Φ a cell-like map.

To verify conclusion (i), note that any map $\mu\colon R \to P_k$ can be approximately lifted, successively, to maps $\mu_i\colon R \to P_k^{(i)}$, $i = 1, 2, \dots, k$, and, finally, to $\mu_{k+1}\colon R \to \widetilde{P_k} \subset M(\phi_{k+1,k}^{-1}(P_k^{(k)})) \subset X_{k+1}$. According to Lemma 3.3, $\Phi_{k+1}^{-1}(M(X_{k+1}))$ is a 3-manifold (where Φ_{k+1} satisfies $\Phi = \phi_{k+1,0}\Phi_{k+1}$). Hence, μ has approximate lifts to $M(Y)$.

To verify conclusion (ii), let A_0 denote $\{p\}$ and recursively let A_n denote $\phi_{n,n-1}^{-1}(A_{n-1}) - M(X_n)$ for $n \in \mathbb{N}$. Each set A_n is finite, by conclusion (4) of Lemma 5.3. Furthermore, $\Phi^{-1}(p) \cap S(Y) \subset A_\infty = \varprojlim A_n$. But the inverse limit of finite sets is 0-dimensional. □

Corollary 5.6. *A 3-gm X has a near-resolution if there exist a sequence $\{P_i\}_{i=1}^\infty$ of preferred 2-polyhedra in X and a family of maps $\{\mu_i\colon R_i \to X\}_{i=1}^\infty$ satisfying the following conditions:*

(i) *$\mu_i(R_i) \subset P_i$ for every $i \geq 1$,*

(ii) *$P_i \subset P_{i+1}$ for every $i \geq 1$, and*

(iii) *given distinct points $p, q \in X$ there exists $k \in \mathbb{N}$ such that μ_k homologically separates p from q.*

Proof. Applying Resolution Lemma 5.5 to X and $\{P_i\}_{i=1}^\infty$, we obtain $\Phi\colon Y \to X$ such that, for all $x \in X$, $S(Y) \cap \Phi^{-1}(x)$ is 0-dimensional. We will show that $\dim S(Y) \leq 0$, which will imply that Y has a near resolution $\psi\colon M \to Y$. The near-resolution of X then will be $\Phi\psi\colon M \to X$.

To show that $\dim S(Y) \leq 0$, we first establish the following

Claim. For any two distinct points p and q of X there exists a map $\kappa\colon R \to Y$ defined on a compact 2-polyhedron R such that $\kappa(R)$ strongly separates $\Phi^{-1}(p)$ from $\Phi^{-1}(q)$ and $\kappa(R) \subset M(Y)$.

Proof of the Claim. Choose $i \in \mathbb{N}$ such that μ_i homologically separates p and q in X. Endow X with a metric, and choose $\epsilon > 0$ such that any ϵ-approximation to μ_i is homotopic to μ_i in $X - \{p, q\}$. By Resolution Lemma 5.5 μ_i has an ϵ-lift κ to $M(Y)$. Since $\Phi\kappa$ is homotopic to μ_i in $X - \{p, q\}$ and Φ restricts to a proper homotopy equivalence of the pairs

$$(Y, Y - \Phi^{-1}(\{p, q\})) \to (X, X - \{p, q\}),$$

it follows that κ homologically separates $\Phi^{-1}(p)$ and $\Phi^{-1}(q)$. By Lemma 3.5, κ strongly separates $\Phi^{-1}(p)$ and $\Phi^{-1}(q)$.

Given a component C of $S(Y)$, one can immediately produce an $x_C \in X$ for which $C \subset \Phi^{-1}(x_C)$, using the Claim. Since $C \subset S(Y) \cap \Phi^{-1}(x_C)$ and since $\dim[S(Y) \cap \Phi^{-1}(x_C)] \leq 0$ by conclusion (ii) of Lemma 5.5, C must be a singleton. Hence, $\dim S(Y) \leq 0$. □

Proof of Theorem 5.1. Apply Lemma 5.4 to obtain a sequence $\{(K_i, P_i)\}_{i\geq 1}$ of compact, tame-preferred polyhedral pairs such that $P_i \subset P_{i+1}$ for all $i \geq 1$ and any two points of X are homologically separated by some map $\mu\colon R \to P_k$ into one of these P_i. Corollary 5.6 assures that X has a near-resolution. □

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Department of Mathematics, The University of Tennessee at Knoxville, Knoxville, Tennessee 37996-1300
E-mail address: `daverman@math.utk.edu`

Department of Mathematics, Texas State University, San Marcos, Texas 78666
E-mail address: `tt04@txstate.edu`